1089. *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.*

Prove that if $x, y, z \geq 1$, then

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z} \geq \frac{1}{1+\sqrt{xy}}+\frac{1}{1+\sqrt{yz}}+\frac{1}{1+\sqrt{zx}} \geq \frac{3}{1+\sqrt[3]{xyz}}.$$

1090. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.*

Find all nonconstant, differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the functional equation $f(x+y) - f(x-y) = 2f'(x)f'(y)$ for all $x, y \in \mathbb{R}$.

SOLUTIONS

An inequality between Hölder and Lehmer means

1061. *Proposed by Arkday Alt, San Jose, CA.*

Let $m \geq n \geq 2$ be positive integers. Prove that for any positive real numbers a and b,

$$\left(\frac{a^m+b^m}{a^{m-1}+b^{m-1}}\right)^{n+1} \geq \frac{a^{n+1}+b^{n+1}}{2}.$$

Solution by Northwestern University Math Problem Solving Group.

The inequality is equivalent to

$$L_m(a,b) \geq M_{n+1}(a,b)$$

for integers $m \geq n \geq 2$, where $L_p(a,b)$ is the p-th Lehmer mean of a and b and $M_p(a,b)$ is the p-th power mean of a and b:

$$L_p(a,b) = \frac{a^p+b^p}{a^{p-1}+b^{p-1}}, \quad M_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}.$$

Here, we will prove the stronger result that, if p and q are real numbers such that $p \geq 1$ and $q \geq \frac{p+1}{2}$, then

$$L_q(a,b) \geq M_p(a,b).$$

Note that the original inequality follows from this result by letting $p = n+1$, $q = m$, and showing $m \geq n \geq 2$ implies $p \geq 1$ and $q \geq \frac{p+1}{2}$.

For the proof, note that $L_q(a,b)$ is increasing in q. This can be proved by logarithmic differentiation:

$$\begin{aligned}\frac{d}{dq}\log L_q(a,b) &= \frac{a^q\log a + b^q \log b}{a^q+b^q} - \frac{a^{q-1}\log a + b^{q-1}\log b}{a^{q-1}+b^{q-1}} \\ &= \frac{a^{q-1}b^{q-1}(a-b)(\log a - \log b)}{(a^q+b^q)(a^{q-1}+b^{q-1})} \geq 0.\end{aligned}$$

Hence, all we need to prove is $L_{\frac{p+1}{2}}(a,b) \geq M_p(a,b)$ for $p \geq 1$.

Assume without loss of generality that $a \le b$ and let $x = b/a$. After dividing both sides of the inequality by a, it becomes $L_{\frac{p+1}{2}}(1, x) \ge M_p(1, x)$ with $x \ge 1$. For $x = 1$, the inequality becomes equality, so all we need to prove is that $L_{\frac{p+1}{2}}(1, x)$ grows faster than $M_p(1, x)$ as $x \ge 1$ increases. To that end, we compute

$$\frac{d}{dx}\left(\log L_{\frac{p+1}{2}}(1, x) - \log M_p(1, x)\right) = \frac{\frac{p+1}{2}x^{\frac{p-1}{2}}}{1 + x^{\frac{p+1}{2}}} - \frac{\frac{p-1}{2}x^{\frac{p-3}{2}}}{1 + x^{\frac{p-1}{2}}} - \frac{x^{p-1}}{1 + x^p}$$

$$= \frac{x^{\frac{p-3}{2}}[(p-1)x^{p+1} - (p+1)x^p + (p+1)x - p + 1]}{2\left(1 + x^{\frac{p+1}{2}}\right)\left(1 + x^{\frac{p-1}{2}}\right)(1 + x^p)}.$$

It suffices now to prove that $f_p(x) \ge 0$ for $x \ge 1$ where

$$f_p(x) = (p-1)x^{p+1} - (p+1)x^p + (p+1)x - p + 1.$$

In fact, we have

$$f_p'(x) = (p+1)(p-1)x^p - (p+1)px^{p-1} + p + 1$$

and

$$f_p''(x) = (p+1)p(p-1)x^{p-1} - (p+1)p(p-1)x^{p-2}$$
$$= (p+1)p(p-1)x^{p-2}(x-1).$$

Note that $f_p(1) = f_p'(1) = f_p''(1) = 0$. Also, for $x \ge 1$, we have $f_p''(x) \ge 0$, hence $f_p'(x)$ is nondecreasing, so $f_p'(x) \ge 0$ for $x \ge 1$, which implies that $f_p(x)$ is also nondecreasing, and $f_p(x) \ge 0$ for $x \ge 1$.

Extension. Note that $f_p''(x) \ge 0$ for $x \ge 1$ if $p \in (-1, 0) \cup (1, \infty)$, and $f_p''(x) \le 0$ for $x \ge 1$ if $p \in (-\infty, -1) \cup (0, 1)$. So by the reasoning above, we have the following more general results.

1. If $p \in (-1, 0) \cup (1, \infty)$, then $L_{\frac{p+1}{2}}(a, b) \ge M_p(a, b)$. The inequality becomes equality precisely for $a = b$.
2. If $p \in (-\infty, -1) \cup (0, 1)$, then $L_{\frac{p+1}{2}}(a, b) \le M_p(a, b)$. The inequality becomes equality precisely for $a = b$.
3. For $p \in \{-\infty, -1, 0, 1, \infty\}$, the equality becomes an identity:
   - $L_{-\infty}(a, b) = M_{-\infty}(a, b) = \min\{a, b\}$,
   - $L_0(a, b) = M_{-1}(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}$, the harmonic mean of a and b,
   - $L_{1/2}(a, b) = M_0(a, b) = \sqrt{ab}$, the geometric mean of a and b,
   - $L_1(a, b) = M_1(a, b) = \frac{a+b}{2}$, the arithmetic mean of a and b,
   - $L_\infty(a, b) = M_\infty(a, b) = \max\{a, b\}$, respectively.

Also solved by ROBERT AGNEW, Buffalo Grove, IL, and Palm Coast, FL; MICHEL BATAILLE, Rouen, France; BRIAN BRADIE, Christopher Newport U.; HONGWEI CHEN, Christopher Newport U.; JOHN CHRISTOPHER, California State U. Sacramento; EUGENE HERMAN, Grinnell C.; PANAGIOTIS KRASOPOULOS, Athens, Greece; ELIAS LAMPAKIS, Kiparissia, Greece; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, U. Rome Tor Vergata, Italy; and the proposer. One incomplete solution was received.